B.sc(math H)part3 paper6
Topic Internal direct product of two groups
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Internal direct product of two groups

Definition : If H, K be any two sub-groups of a group G, then G is said to be internal product of H and K,

i.e., $G = H \times K$,

if (i) Every element of H commutes with every element of K

(ii) Every element of G is uniquely expressible as a product of an element of H and an element of K.

Theorem 4. *If H and K are two sub-group of a group G such that $G = H \times K$, then H and K are normal sub-group of G and* $\frac{G}{H} \cong K$ and $\frac{G}{K} \cong H$. ...(I)

Proof : Let us suppose the mapping $f : G \to H$ defined as

$$\phi(g) = \phi(hk) = h \quad \text{......(1)}$$

where $g = hk$ is any element of G, $G = H \times K$.

Again by definition, h and k are unique elements of H and K respectively. Now we have to prove that ϕ defined as above is a homomorphism of G onto H.

Consider $g_1 = h_1 k_1$ and $g_2 = h_2 k_2$.

$\therefore \quad g_1 g_2 = h_1 k_1 h_2 k_2$

$= h_1 h_2 k_1 k_2$, by definition

$= (h_1 h_2)(k_1 k_2)$, where $h_1 h_2 \in H$ and $k_1 k_2 \in K$.

$\therefore \quad \phi(g_1 g_2) = \phi[(h_1 h_2)(k_1 k_2)]$

$= h_1 h_2$, by definition of ϕ

$= \phi(g_1)\phi(g_2)$.

Also ϕ is obviously onto. Thus ϕ is a homomorphism of G onto H.

Let us now find kernel of $\phi : G \to H$.

Kernel of $\phi = \{g : \phi(g) = e$ the identity of $H\}$

$\Rightarrow g = ek = k \in K$.

$\therefore$ Kernel of $\phi = \{k : k \in K\} = K$.

Since the Kernel of ϕ is a normal sub-group and hence K is a normal sub-group of G.

Also $\frac{G}{K}$ is a quotient group. By fundamental theorem of homomorphism, we know that homomorphic image of H of $\phi : G \to H$ is isomorphic to quotient group $\frac{G}{K}$.

$$\therefore \quad \frac{G}{K} \cong H.$$

Similarly we can prove that H is a normal sub-group and

$$\frac{G}{H} \cong K.$$

Theorem 5. *If a group G be the internal direct product of its sub-groups H and K, then*

(i) H and K have only the identity in common.

(ii) G is isomorphic to external direct product of H by K, i.e. $G \cong H \times K$.

Proof : (i) Consider $x \in H \cap K$ *so that* $x \in H$ and $x \in K$ and

$x^{-1} \in H, x^{-1} \in K$

because both H and K are sub-groups.

Again $g \in G$ can be uniquely expressed as

$g = hk : h \in H, k \in K$ as $G = H \times K$ ......(1)

and $g = (hx)(x^{-1}k) : hx \in H, x^{-1}k \in K$ (2)

By definition of $G = H \times K$, each element of G can be uniquely expressed as the product of an element of H and an element of K.

Thus from (1) and (2), we conclude that

$hx = h$ and $x^{-1} k = k$.

$\therefore \quad hx = h \Rightarrow x = e$.

Therefore e the identity is the only element common to H and K i.e. $e \in H \cap K$.

(ii) Suppose that the mapping $\phi : G \to H$ is defined by

$$\phi(g) = \phi(hk) = (h, k), \forall \; g \in G.$$

ϕ is one-one : Let $\phi(g_1) = \phi(g_2) \Rightarrow (h_1, k_1) = (h_2, k_2)$

$\Rightarrow \; h_1 = h_2$ and $k_1 = k_2$

$\Rightarrow \; h_1 k_1 = h_2 k_2$

$\Rightarrow \; g_1 = g_2$

$\Rightarrow \; \phi$ is one-one.

ϕ is onto : Suppose (h, k) be any element of $H \times K$, then $hk \in G$. Also $\phi(hk) = (h, k)$, by definition.

$\Rightarrow \; \phi$ is onto.

ϕ preserves the composition :

$$\phi(g_1 g_2) = \phi(h_1 k_1, h_2 k_2) = \phi(h_1 h_2 k_1 k_2)$$

because every element of H commutes with every element of K.

$$\therefore \quad \phi(g_1 g_2) = (h_1 h_2, k_1 k_2), \quad (\because h_1 h_2 \in H, \; k_1 k_2 \in K)$$
$$= (h_1, k_1)(h_2, k_2)$$
$$= \phi(h_1 k_1) \phi(h_2 k_2)$$
$$= \phi(g_1) \phi(g_2).$$

$\because \; \phi : G \to H \times K$ is one-one onto and preserves the group composition therefore it is an isomorphism.

Theorem 6. *A group G is the direct product of its two sub-groups H and K i.e., G = H × K, if and only if*

(i) H and K are normal sub-groups,

(ii) $H \cap K = \{e\}$,

(iii) G = HK.

Proof : Let us suppose that the conditions (i), (ii) and (iii) holds good. Then we have to prove that $G = H \times K$, *i.e.*, conditions (i) and (ii) of definition of internal direct product hold good.

Consider $h \in H$, $k \in K$ so that $h^{-1} \in H$ and $k^{-1} \in K$.

Suppose the element $h^{-1}k^{-1}hk = h^{-1}(k^{-1}hk)$

$= h^{-1}$ (some element of H as H is a normal sub-group)

$= H$, as $h^{-1} \in H$ and $a \in H \Rightarrow h^{-1} a \in H$.

Again $h^{-1}k^{-1}hk = (h^{-1}k^{-1}h)k$

= (some element b of K as K is a normal sub-group) k

$= bk \in K$.

Since $h^{-1}k^{-1}hk$ belongs to both H and K, therefore

$h^{-1}k^{-1}hk$ belongs to $H \cap K = \{e\}$, by (ii).

$\therefore \quad (h^{-1}k^{-1})hk = e$

or, $(kh)^{-1}(hk) = e \Rightarrow hk = (kh)^{-1})^{-1} = kh$.

This shows that every element of H commutes with every element of K which is condition (i) of definition of $G = H \times K$.

Suppose $g \in G$ *be expressed as* $g = hk : h \in H, k \in K$.

Now we have to prove that the condition (ii) of definition of $G = H \times K$ i.e. the above expression for g is unique.

If possible suppose $g = h_1k_1 : h_1 \in H_1, k_1 \in K$.

$\therefore \quad hk = h_1k_1 \Rightarrow h_1^{-1}(hk)k^{-1} = h_1^{-1}(h_1k_1)k^{-1}$

$\Rightarrow (h_1^{-1}h)(kk^{-1}) = (h_1^{-1}h_1)(k_1k^{-1})$

$\Rightarrow h_1^{-1}h = k_1k^{-1}$,

Since $h_1^{-1}h \in H$ and $k_1k^{-1} \in K$ therefore each of them belongs to $H \cap K = \{e\}$ by (ii).

$\therefore \quad h_1^{-1}h = e$ and $k_1k^{-1} = e$ or $h_1 = h$ and $k_1 = k$.

Hence the expression for $g = hk$ is unique

$\therefore \quad G = H \times K$.

Conversely : Let $G = H \times K$ so that $g = hk$ subject to conditions (i) and (ii) of definition of $G = H \times K$ and we will prove the conditions (i), (ii) and (iii) of this theorem.

(i) H and K are normal sub groups.

Let a be any element of H and $g = hk$ be any element of G.

$g^{-1}ag = (hk)^{-1}a(hk) = (k^{-1}h^{-1})a(hk)$

$= (h^{-1}k^{-1})a(kh)$ by condition (i) of definition

$= h^{-1}k^{-1}(ak)h$

$= h^{-1}(k^{-1}ka)h = h^{-1}ah \in H$.

Since for every $g \in G$ and $a \in H$, we have $g^{-1}\,ag \in H$,

Therefore H is normal sub group of G

Similarly K is also a normal sub-group of G.

(ii) Now we shall prove that $H \cap K = \{e\}$.

Let $a \neq e$ be any element of $H \cap K$ so that $a \in H$, $a \in K$. ar certainly $a \in G$.

If e be the identity which belongs to both H and K and is the same that of G.

Now $a = ea = ae$.

This shows that an element a of G is expressible in two different wa as the product of an element of H and that of K.

But this violates the definition (ii) of $G = H \times K$.

Hence our supposition that $a \neq e$ is wrong and as such

$$H \cap K = \{e\}.$$

(iii) Now we will show that $G = HK$.

We know that $HK \subseteq G$.

Let $g \in G$ so that $g = hk$, $h \in H$, $k \in K$.

$$\Rightarrow g \in HK \Rightarrow G \subseteq HK.$$

since $HK \subseteq G$ and $G \subseteq HK$ therefore it follows that

$$G = HK.$$